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Abstract. A static exact solution of Einstein's field equations is obtained when the region of space–time consists of a coupled electromagnetic and zero-mass scalar meson field. This solution has been built up from the 'Curzon' particle solution. The solution does not represent directional singularities as the origin is approached contrary to what has been established in the case of pure scalar fields by Gautreau.

1. Introduction

The coupled electromagnetic and scalar meson fields in general relativity have been studied by various authors. Stephenson (1962) has solved the field equations of general relativity when the region of the space–time consists of coupled electromagnetic and nonzero-rest-mass scalar meson fields for the static spherically symmetric metric and has obtained an approximate solution. Roy and Rao (1972) have shown that in the axially symmetric case the massive scalar meson field cannot be the source of a gravitational field. As such, in a separate investigation Rao *et al* (1972) have taken up the problem of coupled electromagnetic and zero-mass scalar meson fields and obtained a class of exact solutions. Some of these solutions show that the presence of zero-mass scalar meson fields does not affect the singular behaviour as when only an electromagnetic field is present. Gautreau (1969) has investigated the interacting gravitational and zero-mass scalar fields with the 'Curzon' solution (Curzon 1924) as the corresponding solution for the empty-space field equations. It has been observed by him that the addition of a zero-mass scalar meson field does not affect the singular behaviour as when compared to the purely 'Curzon' gravitational field. In the present paper, we have extended the above study of Gautreau to the case when the space–time region contains in addition to the zero-mass scalar field, a source-free electromagnetic field. In § 2, the field equations for the coupled fields have been obtained. Section 3, deals with a brief review of the method obtained by Janis *et al* (1969) for developing the coupled-field solutions for electromagnetic and scalar meson fields from the vacuum solutions. This technique has been used to obtain the coupled-field solutions for a static axially-symmetric field in § 4. In the concluding section, we have investigated the singular behaviour of the solutions which shows that the solution is singular irrespective of the direction from which the origin is approached. This result is contrary to the behaviour of the vacuum-field solution and purely scalar-field solution as established by Gautreau and Anderson (1967), and Gautreau (1969).

2. The field equations

We consider the static cylindrically symmetric metric

$$ds^2 = -e^{2\nu-2\lambda}(dr^2+dz^2)-r^2 e^{-2\lambda} d\phi^2+e^{2\lambda} dt^2, \tag{1}$$

the signature of the metric being $(-1, -1, -1, +1)$ and λ and ν are functions of r and z only.

The field equations of general relativity when the space–time region contains an electromagnetic field satisfying Maxwell's equations

$$F_{ij} = A_{i,j}-A_{j,i}, \tag{2}$$

$$F^{ij}{}_{;j} = 0 \tag{3}$$

and a zero-mass scalar meson field satisfying

$$g^{ij}V_{;ij} = 0, \tag{4}$$

are given by

$$G^{\mu}_{\nu} \equiv R^{\mu}_{\nu}-\tfrac{1}{2}\delta^{\mu}_{\nu}R = -\kappa(T^{\mu(s)}_{\nu}+T^{\mu(e)}_{\nu}), \tag{5}$$

where κ $(= 8\pi G/c^4)$ is the gravitational constant. $T^{\mu(s)}_{\nu}$ is the energy–momentum tensor for the zero-mass scalar field given by

$$T^{\mu(s)}_{\nu} = \frac{1}{4\pi}(V^{,\mu}V_{,\nu}-\tfrac{1}{2}\delta^{\mu}_{\nu}V_{,s}V_{,p}g^{sp}) \tag{6}$$

and $T^{\mu(e)}_{\nu}$ is the energy–momentum tensor for the electromagnetic field given by

$$T^{\mu(e)}_{\nu} = \frac{1}{4\pi}(-F_{\nu s}F^{\mu s}+\tfrac{1}{4}\delta^{\mu}_{\nu}F_{sp}F^{sp}). \tag{7}$$

The field equations (5) for the metric (1) reduce to

$$\lambda^2_{,1}-\lambda^2_{,2}-\frac{\nu_{,1}}{r} = -\frac{\kappa}{8\pi}(V^2_{,1}-V^2_{,2})+e^{-2\lambda}\frac{\kappa}{8\pi}\left(\frac{1}{r^2}e^{4\lambda}F^2_{23}-F^2_{24}-\frac{1}{r^2}e^{2\nu}F^2_{34}\right.$$
$$\left.-e^{4\lambda-2\nu}F^2_{12}-\frac{1}{r^2}e^{4\nu}F^2_{13}+F^2_{14}\right), \tag{8}$$

$$-\lambda^2_{,1}+\lambda^2_{,2}+\frac{\nu_{,1}}{r}$$
$$= \frac{\kappa}{8\pi}(V^2_{,1}-V^2_{,2})-e^{-2\lambda}\left(\frac{\kappa}{8\pi}\right)\left(F^2_{14}-\frac{1}{r^2}e^{4\lambda}F^2_{13}+\frac{1}{r^2}e^{2\nu}F^2_{34}\right.$$
$$\left.+e^{4\lambda-2\nu}F^2_{12}+\frac{1}{r^2}e^{4\lambda}F^2_{23}-F^2_{24}\right), \tag{9}$$

$$-\nu_{,11}-\nu_{,22}-\lambda^2_{,1}-\lambda^2_{,2}$$
$$= \left(\frac{\kappa}{8\pi}\right)(V^2_{,1}+V^2_{,2})-e^{-2\lambda}\frac{\kappa}{8\pi}\left(F^2_{14}+F^2_{24}-e^{4\lambda-2\nu}F^2_{12}\right.$$
$$\left.+\frac{1}{r^2}e^{4\lambda}F^2_{13}+\frac{1}{r^2}e^{4\lambda}F^2_{23}-\frac{1}{r^2}e^{2\nu}F^2_{34}\right), \tag{10}$$

$$-\nu_{,11}-\nu_{,22}-\lambda_{,1}^2-\lambda_{,2}^2+2\left(\lambda_{,11}+\lambda_{,22}+\frac{\lambda_{,1}}{r}\right)$$
$$=\frac{\kappa}{8\pi}(V_{,1}^2+V_{,2}^2)+e^{-2\lambda}\frac{\kappa}{8\pi}\left(e^{4\lambda-2\nu}F_{12}^2+\frac{1}{r^2}e^{4\lambda}F_{13}^2+\frac{1}{r^2}e^{4\lambda}F_{23}^2\right.$$
$$\left.+F_{14}^2+F_{24}^2+\frac{e^{2\nu}}{r^2}F_{34}^2\right), \tag{11}$$

$$2\lambda_{,1}\lambda_{,2}-\frac{\nu_{,2}}{r}=-\frac{\kappa}{8\pi}(2V_{,1}V_{,2})+e^{-2\lambda}\frac{\kappa}{8\pi}\left(\frac{2}{r^2}e^{4\lambda}F_{23}F_{13}+2F_{24}F_{14}\right) \tag{12}$$

The scalar wave equation (4) gives

$$\Box V\equiv e^{-2\nu+2\lambda}\left(V_{,11}+V_{,22}+\frac{V_{,1}}{r}\right)=0. \tag{13}$$

3. Technique for developing the solutions

Janis *et al* (1969) have obtained a result which expresses the solution of the coupled electromagnetic and zero-mass scalar fields in terms of the known solution of the corresponding empty-space field equations.

If the metric of the line element

$$ds^2=e^{2v}(dx^4)^2-e^{-2v}h_{ij}\,dx^i\,dx^j, \tag{14}$$

represents the empty-space solutions of Einstein's field equations then a static solution of the Einstein scalar-field equations is given by the metric of the line element

$$ds^2=e^{2u}(dx^4)^2-e^{-2u}h_{ij}\,dx^i\,dx^j \tag{15}$$

and V, where

$$V=Au \tag{16}$$

and

$$u=v\left(1+\frac{\kappa A^2}{8\pi}\right)^{-1/2}, \tag{17}$$

A being a constant. If the region of space–time contains a source-free electromagnetic field in addition to the zero-mass scalar field then the solution of the Einstein–Maxwell scalar field equations

$$G_j^i\equiv R_j^i-\tfrac{1}{2}\delta_j^iR=-\frac{\kappa}{4\pi}(V^{,i}V_{,j}-\tfrac{1}{2}\delta_j^ig^{sp}V_{,s}V_{,p}-F_{js}F^{is}+\tfrac{1}{4}\delta_j^iF_{sp}F^{sp}), \tag{18}$$

is given by

$$V, F_{ij}=\left(\frac{8\pi}{\kappa}\right)^{1/2}e^{2w}(\delta_i^4u_{,j}-\delta_j^4u_{,i}) \tag{19}$$

and the metric for the line element

$$ds^2=e^{2w}(dx^4)^2-e^{-2w}h_{ij}\,dx^i\,dx^j, \tag{20}$$

where

$$w = -\ln \sinh u. \tag{21}$$

4. Solution of the coupled fields

In this section, we shall generate the solutions of the field equations (8) to (13) for the coupled electromagnetic and zero-mass scalar fields by applying the result obtained in the previous section.

We consider the empty-space solution to be given by the well known 'Curzon' solution

$$\lambda = -\frac{m}{\rho},$$

$$\nu = -\frac{m^2 r^2}{2\rho^4}, \tag{22}$$

where

$$\rho = (r^2 + z^2)^{1/2}.$$

On substituting these values of λ and ν in (1) the metric can be written as

$$ds^2 = \exp\left(-\frac{2m}{\rho}\right) dt^2 - \exp\left(\frac{2m}{\rho}\right)\left\{\exp\left(-\frac{m^2 r^2}{2\rho^4}\right)(dr^2 + dz^2) + d\phi^2\right\}. \tag{23}$$

Identifying (23) and (14), we get

$$v = -\frac{m}{\rho}. \tag{24}$$

Now from (17), we have

$$u = -\frac{m}{\rho}\left(1 + \frac{\kappa A^2}{8\pi}\right)^{-1/2}. \tag{25}$$

On substituting this value of u in (15), we get

$$ds^2 = \exp\left\{-\frac{2m}{\rho}\left(1 + \frac{\kappa A^2}{8\pi}\right)^{-1/2}\right\} dt^2 - \exp\left\{\frac{2m}{\rho}\left(1 + \frac{\kappa A^2}{8\pi}\right)^{-1/2}\right\}$$
$$\times \left\{\exp\left(-\frac{m^2 r^2}{2\rho^4}\right)(dr^2 + dz^2) + d\phi^2\right\}. \tag{26}$$

Identifying (26) and (1), we have

$$\lambda = -\frac{m}{\rho}\left(1 + \frac{\kappa A^2}{8\pi}\right)^{-1/2},$$

$$\nu = -\frac{m^2 r^2}{2\rho^4}.$$

Hence, the solution of the Einstein scalar-field equation is given by

$$\lambda = -\frac{m}{\rho}\left(1+\frac{\kappa A^2}{8\pi}\right)^{-1/2},$$
$$\nu = -\frac{m^2 r^2}{2\rho^4} \tag{27}$$

and the scalar potential

$$V = -\frac{Am}{\rho}\left(1+\frac{\kappa A^2}{8\pi}\right)^{-1/2}.$$

From (21), we have for the coupled electromagnetic and zero-rest-mass field

$$\begin{aligned} w &= -\ln\sinh u \\ &= -\ln\sinh\left\{-\frac{m}{\rho}\left(1+\frac{\kappa A^2}{8\pi}\right)^{-1/2}\right\} \\ &= \ln\operatorname{cosech}\left(-\frac{mA'}{\rho}\right), \end{aligned}$$

where A' $(= (1+\kappa A^2/8\pi)^{-1/2})$ is a constant.

On substituting this value of w in (20), we have

$$ds^2 = \operatorname{cosech}^2\left(-\frac{mA'}{\rho}\right) dt^2 - \sinh^2\left(-\frac{mA'}{\rho}\right)\left\{\exp\left(-\frac{m^2r^2}{2\rho^4}\right)(dr^2 + dz^2) + d\phi^2\right\}. \tag{28}$$

Identifying (28) and (1), we get the solution corresponding to coupled electromagnetic and zero-mass scalar fields as

$$\lambda = \ln\operatorname{cosech}\left(-\frac{mA'}{\rho}\right),$$
$$\nu = -\frac{m^2r^2}{2\rho^4}.$$

From the equation (19), it is clear that the only nonzero components of the electromagnetic field tensor are F_{14} and F_{24}. Thus, with the help of (19), we have

$$F_{14} = -\left(\frac{8\pi}{\kappa}\right)^{1/2} e^{2w}\frac{\partial}{\partial r}\left(-\frac{mA'}{(r^2+z^2)^{1/2}}\right) = -\left(\frac{8\pi}{\kappa}\right)^{1/2} mA'r(r^2+z^2)^{-3/2}\operatorname{cosech}^2\left(-\frac{mA'}{\rho}\right)$$

and

$$F_{24} = -\left(\frac{8\pi}{\kappa}\right)^{1/2} mA'z(r^2+z^2)^{-3/2}\operatorname{cosech}^2\left(-\frac{mA'}{\rho}\right).$$

Hence, a solution of the coupled field is given by

$$V = -\frac{A''m}{\rho},$$

where $A''(=AA')$ is a constant,

$$
\begin{aligned}
F_{14} &= -\left(\frac{8\pi}{\kappa}\right)^{1/2} mA'\, r(r^2+z^2)^{-3/2}\,\mathrm{cosech}^2\left(-\frac{mA'}{\rho}\right),\\
F_{24} &= -\left(\frac{8\pi}{\kappa}\right)^{1/2} mA'\, z(r^2+z^2)^{-3/2}\,\mathrm{cosech}^2\left(-\frac{mA'}{\rho}\right),\\
\lambda &= \ln\,\mathrm{cosech}\left(-\frac{mA'}{\rho}\right),\\
\nu &= -\frac{m^2r^2}{2\rho^4}.
\end{aligned}
\tag{29}
$$

5. Conclusions

The Kretschmann curvature invariant

$$\alpha = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$

is evaluated for the solution (29) and obtained in the form

$$
\begin{aligned}
\alpha = e^{-4\nu+4\lambda}\Bigg\{&\{\nu_{,22}+\nu_{,11}-(\lambda_{,22}+\lambda_{,11})\}^2+\left(\lambda_{,11}-\lambda_{,12}^2-\lambda_{,1}\nu_{,1}+\lambda_{,2}\nu_{,2}+\frac{1}{r}(\nu_{,1}+\lambda_{,1}\right)^2\\
&+(-\lambda_{,11}-2\lambda_{,1}^2+\lambda_{,1}\nu_{,1}-\lambda_{,2}\nu_{,2}+\lambda_{,2}^2)^2+\Bigg(\lambda_{,22}+\lambda_{,1}\nu_{,1}-\lambda_{,2}\nu_{,2}-\lambda_{,1}^2\\
&+\frac{1}{r}(\lambda_{,1}-\nu_{,1})\Bigg)^2+(-\lambda_{,22}+\lambda_{,1}^2-2\lambda_{,2}^2-\lambda_{,1}\nu_{,1}+\lambda_{,2}\nu_{,2})^2+\Bigg(\lambda_{,12}+\lambda_{,1}\lambda_{,2}\\
&-\lambda_{,1}\nu_{,2}-\lambda_{,2}\nu_{,1}+\frac{\nu_{,2}}{r}\Bigg)^2+(-\lambda_{,12}-3\lambda_{,1}\lambda_{,2}+\lambda_{,1}\nu_{,2}+\lambda_{,2}\nu_{,1})^2\\
&+\left(\lambda_{,1}^2+\lambda_{,2}^2-\frac{\lambda_{,1}}{r}\right)^2\Bigg\}.
\end{aligned}
\tag{30}
$$

The first term in the large curly bracket of the right hand side of (30), yields

$$
\begin{aligned}
&\{\nu_{,22}+\nu_{,11}-(\lambda_{,22}+\lambda_{,11})\}^2\\
&\quad= m^2A'(r^2+z^2)^{-3}\coth^2\left(-\frac{mA'}{\rho}\right)+m^4A'^4(r^2+z^2)^{-4}\,\mathrm{cosech}^4\left(-\frac{mA'}{\rho}\right)\\
&\quad+2m^3A'^3(r^2+z^2)^{-7/2}\,\mathrm{cosech}^2\left(-\frac{mA'}{\rho}\right)\coth\left(-\frac{mA'}{\rho}\right)+2m^3A'(r^2+z^2)^{-7/2}\\
&\quad\times\coth\left(-\frac{mA'}{\rho}\right)+2m^4A'^2(r^2+z^2)^{-4}\,\mathrm{cosech}^2\left(-\frac{mA'}{\rho}\right)+m^4(r^2+z^2)^{-4}.
\end{aligned}
\tag{31}
$$

It may be verified that in (31), terms similar to the last term are present in other brackets also. These do not cancel with each other. We observe that these terms tend to infinity irrespective of the direction from which the origin is approached. Thus we conclude that $\alpha \to \infty$ as $\rho(=(r^2+z^2)^{1/2}) \to 0$ irrespective of direction. This result is contrary to the behaviour of the 'Curzon' solution and the purely scalar-field solution as established by Gautreau and Anderson (1967) and Gautreau (1969).

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